Math 206B Lecture 16 Notes

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1 Hook Walks and Representation Theory of Stanley's Formula

1.1 Hook walks

We want to prove the hook length formula in a more elegant way. This is a construction due to Greene, Nijenhuis, and Wilf in 1979.[1]

We will probabilistically construct a standard Young tableau. First, take the empty diagram. We need to put n into one of the corner spaces. We get

$$\mathbb{P}(n \text{ is at } (r,s)) = \frac{f^{\lambda^*}}{f^\lambda} \overset{\text{WTS}}{=} \frac{(n-1)!}{n!} \prod \frac{h_{i,j}}{h^*_{i,j}},$$

where λ^* is $\lambda - (r,s)$. Using this, if we had a guess for the hook length formula, we could try to prove the hook length formula by induction. However, this is actually not that easy. So GNW constructed a random process to do this.

The idea is a **hook walk**.

1. Start at a random $(i,j) \in \lambda$.
2. Mover to a random square in $\mathrm{Hook}_\lambda(i,j)$.
3. Repeat step 2 until you get to the corner.

Lemma 1.1. *$\mathbb{P}$(corner is at (r,s)) $= f^{\lambda^*}/f^\lambda$.*

Given this lemma, using the fact that $\sum \mathbb{P}(\text{corner is at } (r,s)) = 1$, we get $f^\lambda = \sum_{\lambda^*} f^{\lambda^*}$. This lets us use the guess for the hook length formula to use our induction. Here is a stronger result, which is easier to prove.

[1] This is actually before the NPS algorithm, which is from 1992.

Lemma 1.2. *Let α_i be the column number of the i-th square in this algorithm, and let β_i be the row number of the i-th square. Then*

$$\mathbb{P}((i,j) \to (r,s) \text{ via } \overline{\alpha}, \overline{\beta}) = \frac{1}{n} \prod \left(1 + \frac{1}{h_{i,\alpha} 1}\right) \prod \left(1 + \frac{1}{h_{\beta,j} - 1}\right).$$

For the precise statement of the lemma, look at Exercise 3.17 in the textbook (Sagan's *The Symmetric Group*).

1.2 Stanley's formula, explained

Let $W_n = \mathbb{C}[x_1, \ldots, x_n]$, thought of as an infinite dimensional representation of S_n. Let $H_n \subseteq W_n$ be the set of **harmonic polynomials**, nonconstant polynomials $f \in W_n$ such that $h \cdot f = 0$ for all $h \in \mathbb{C}[\partial/\partial x_1, \ldots, \partial/\partial x_n]^{S_n}$.

Example 1.1. Let $n = 2$. Then what survives when we apply $\partial/\partial x_1, \partial/\partial x_2$ or $\partial/\partial x_1 \partial x_2$? We get that $H_2 = \mathbb{C}\langle 1, x_1 - x_2 \rangle$.

Theorem 1.1 (Chevalley)**.** *Let W_n, H_n be as above.*

1. *$W_n = H_n \otimes I_n$, where $I_n = W^{S_n}$ is the symmetric polynomials.*
2. *H_n is the regular S_n-representation.*

Write $W_n = \bigoplus_{k=0}^{\infty} W_n^k$, $H_n = \bigoplus_{k=0}^{\binom{n}{2}} H_n^k$, and $I_n = \bigoplus_{k=0}^{\infty} I_n^k$. This is a grading by the degree of the polynomials. Write

$$P_n(t) = \sum_{k=0}^{\infty} (\dim(W_n^k) t^k = 1/(1-t)^n$$

$$I_n(t) = \sum_{k=0}^{\infty} (\dim(T_n^k)) t^n = \frac{(1-t)(1-t^2)\cdots(1-t^n)}{.}$$

$$H_n(t) = \frac{P_n(t)}{I_n(t)} = \prod_{i=1}^{n} \frac{1-t^i}{1-t} = \prod_{i=1}^{n} (1 + t + \cdots + t^{i-1}).$$

Now

$$P_\lambda(t) = \sum_{k=0}^{\infty} \dim(\mathrm{Hom}(W_n^k, S^\lambda)) t^k = \prod_{(i,j) \ in \lambda} \frac{t^{i-1}}{1 - t^{h_{i,j}}}.$$

Then

$$H_\lambda(t) = \sum \dim(\mathrm{Hom}(B_n^k, S^\lambda)) t^k = H_n(t) \prod_{(i,j) \in \lambda} \frac{1}{(h_{i,j})_t},$$

the t-analogue of $h_{i,j}$. Then the hook length formula is

$$H_\lambda(1) = f^\lambda = \frac{n!}{\prod h_{i,j}}.$$

This is an algebraic interpretation of how Stanley's formula implies the hook length formula.